<u>Fractals</u>

FRACTURED CURVES:

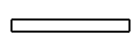
In the world of mathematics, there are “real world” objects that cannot be easily described as polygons, since they are so “irregular” that they cannot be described as being bounded by a finite number of straight line segments. One example of such an irregular object is the coastline of a country. In fact, it is just such an object as the coastline of a country which serves as an example of a newly recognized category of mathematical objects known as *fractal* (The word “fractal” was invented in 1975 by Benoit Mandelbrot.). Figures like these were, until recently, considered “mathematical monstrosities” because they cannot be described by the usual shapes studied in Euclidian geometry. One of the reasons they cannot be so described has to do with the problem we encounter when we try to measure them.

EXERCISE 1: A map of Lake Superior is shown on the next page. Measure the perimeter of the coastline of this lake using each of your two rulers. Record your answers in the chart below

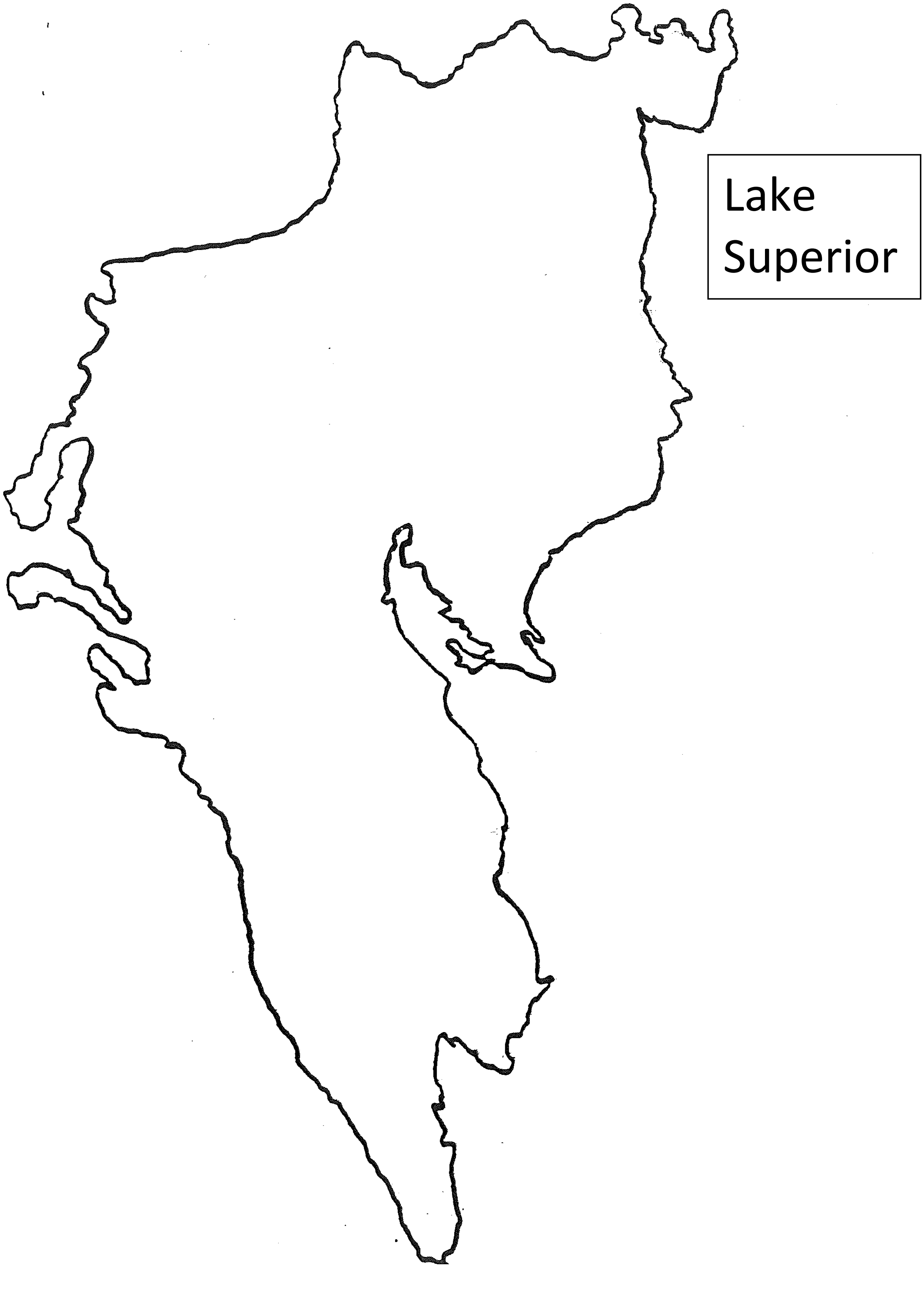
Measurement of Coastline of Lake Superior

	(A)	(B)	(C)
	No. of lengths of ruler 1	No. of lengths of ruler 2	$\frac{\text{Answers in Col. B}}{2}$
Coastline			

Ruler 1



Ruler 2


Lake Superior

- EXERCISE 2: If you used shorter and shorter rulers, what do you think would happen to your measurements? Why?

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- EXERCISE 3: The Portuguese report that the common boundary between Spain and Portugal is over 200 km longer than the Spanish say that it is. Why do you think there is this disagreement? (Hint: Portugal is smaller than Spain, but separate maps of the two countries often show them at approximately the same size as shown below)

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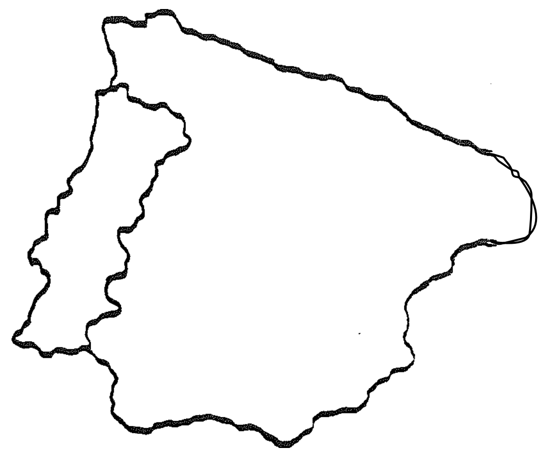
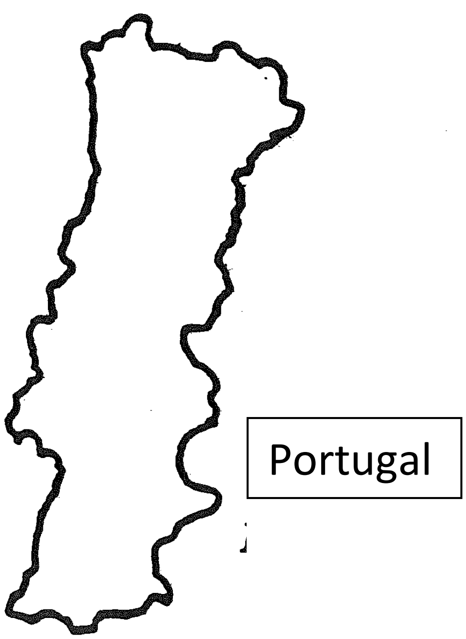
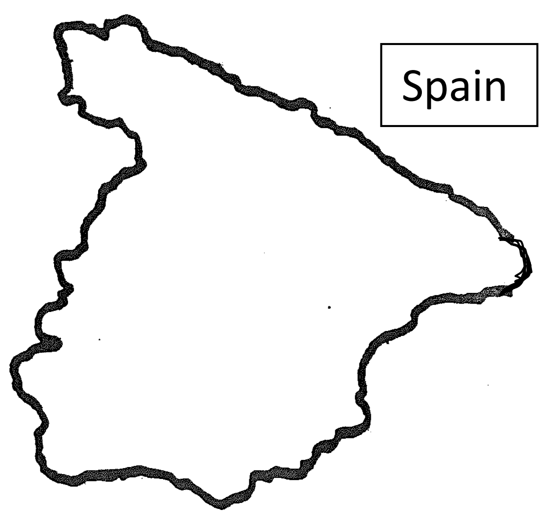






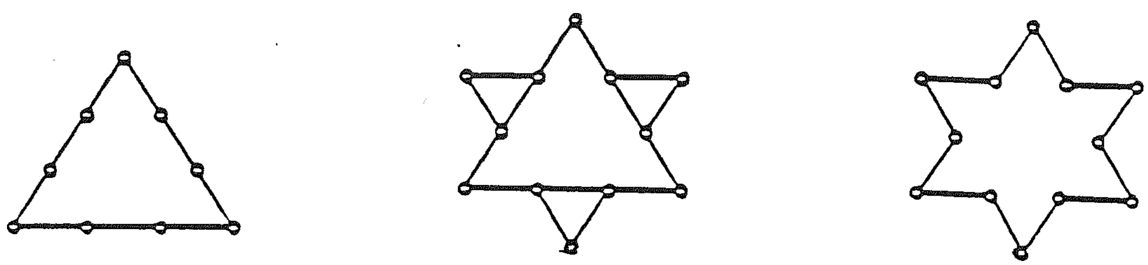
THE KOCH SNOWFLAKE:

To generate this curve, we begin with an equilateral triangle. We will assume that the sides of this triangle are each 1 unit long, so that the perimeter is 3 units. Then, we apply the iterative process below.

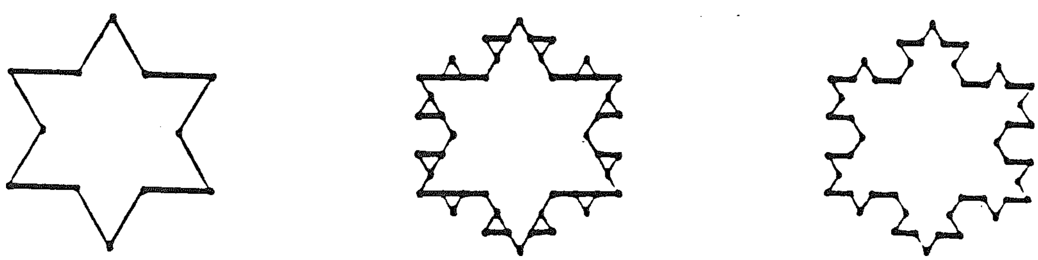
Koch Snowflake Procedure

Stage 1:

a. Trisect each side, i.e., divide each side into 3 equal parts
b. On the middle third of each side, construct an equilateral triangle.
c. Erase the base of each new triangle

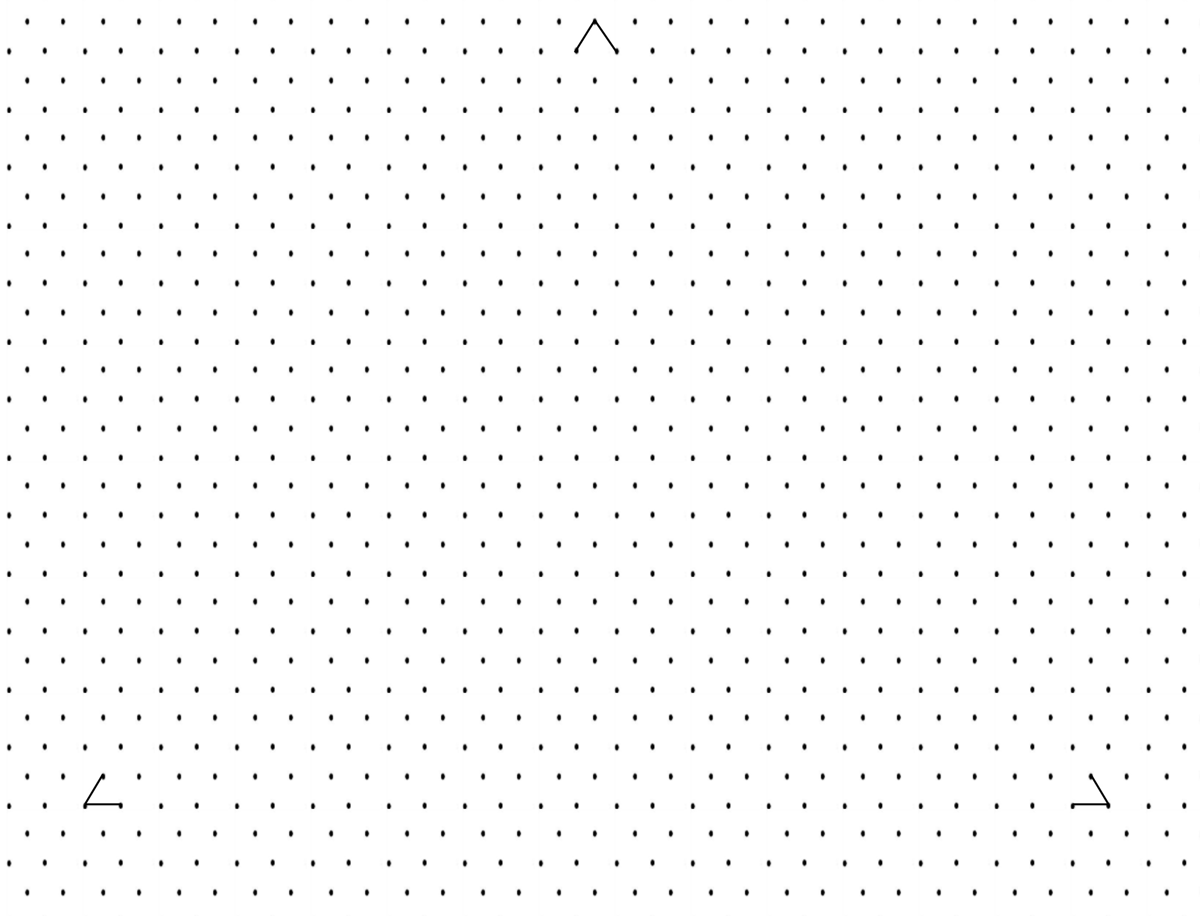


Stage 2: Repeat this procedure on each new side



Stage 3: Continue repeating this procedure. Note that at each stage of the construction, a side or segment is replaced by four new sides or segments, each of which are 1/3 as long

- EXERCISE 4
  A. Use a pencil (so you can erase) to complete the drawing of the equilateral triangle whose vertices are shown below. Then carry out the first three stages in the construction of the Koch Snowflake.
  B. After each stage of your construction, record the perimeter enclosed by your curve in the chart on the next page (your teacher may ask you to record the area also).



Perimeter of Snowflake

	Number of sides	Length of each side	Perimeter	
Stage 0	3	1	3	
Stage 1	12 = 3 (4)			
Stage 2				
Stage 3				
Stage 4				
Stage 5				
Stage 6				
Stage 7				
Stage 10				
Stage 100				
Stage n				

- EXERCISE 5**:** Analyze the patterns which appear in each column in the chart above and fill in the last seven rows of the chart

- EXERCISE 6: What appears to be happening to the perimeter as you continue to iterate this procedure?

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- EXERCISE 7: Since the actual Koch snowflake is the curve obtained when this process is repeated indefinitely, what will be the perimeter of this snowflake?

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- EXERCISE 8: What appears to be happening to the area as you continue to iterate this procedure?

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- EXERCISE 9: Since the actual Koch Snowflake is the curve obtained when this process is repeated indefinitely, what will be the area of this snowflake?

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Class section: ______________ Name: _____________________________

WORKSHEET 2B: GENERATING THE SIERPINSKI GASKET

The Koch Snowflake is an example of a fractal curve. Another fractal curve which is also created from an equilateral triangle is known as the Sierpinski Gasket. But unlike the Snowflake, construction of the gasket involves the interior as well as the boundary of the triangle. The following exercises explore the effects of the construction on an equilateral triangle.

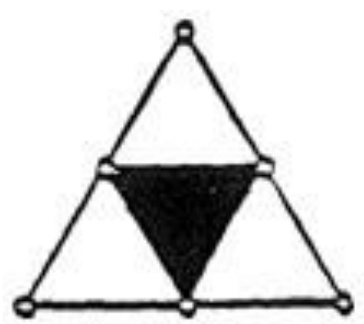
- EXERCISE 1: Let $\triangle$ be a unit equilateral triangle (so each side is a unit long).
  (a) What is the perimeter of $\triangle$? (Hint: sketch the triangle)

  Perimeter = ___________________

  (b) What is the area of $\triangle$?

  Area = __________________

- EXERCISE 2: The construction of the Sierpinski Gasket requires joining the midpoints of the three sides of an equilateral triangle. Verify that the new triangle obtained when the midpoints of an equilateral triangle are joined is also an equilateral triangle



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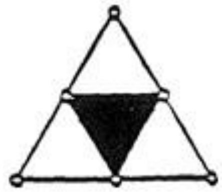
- EXERCISE 3: Let $\triangle$ be a unit equilateral triangle.
  (a) What is the area of the new "middle" triangle created by constructing segments between the midpoints of the three sides of $\triangle$?



Area=______________

  (b) If this new triangle is deleted from $\triangle$, what is the area of the remaining "gasket?"

Area=______________

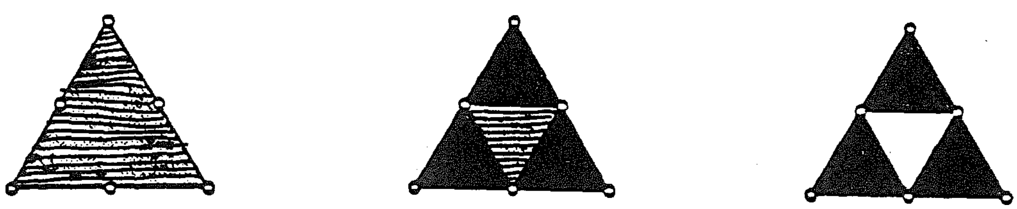
THE SIERPINSKI GASKET:

To generate this figure we begin with an equilateral triangle. We will again assume that the sides of this triangle are each one unit long, so that the perimeter is 3 units. We then apply the iterative procedure below.

Sierpinski Gasket Procedure

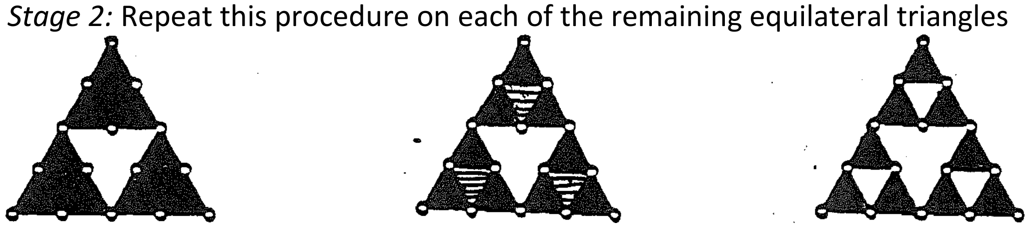
Stage 1:

(a) Bisect each side of an equilateral triangle
(b) Construct segments between the midpoints of the three sides
(c) Remove the interior of the new "middle" triangle



(a) (b) (c)

Stage 2: Repeat this procedure on each of the remaining equilateral triangles



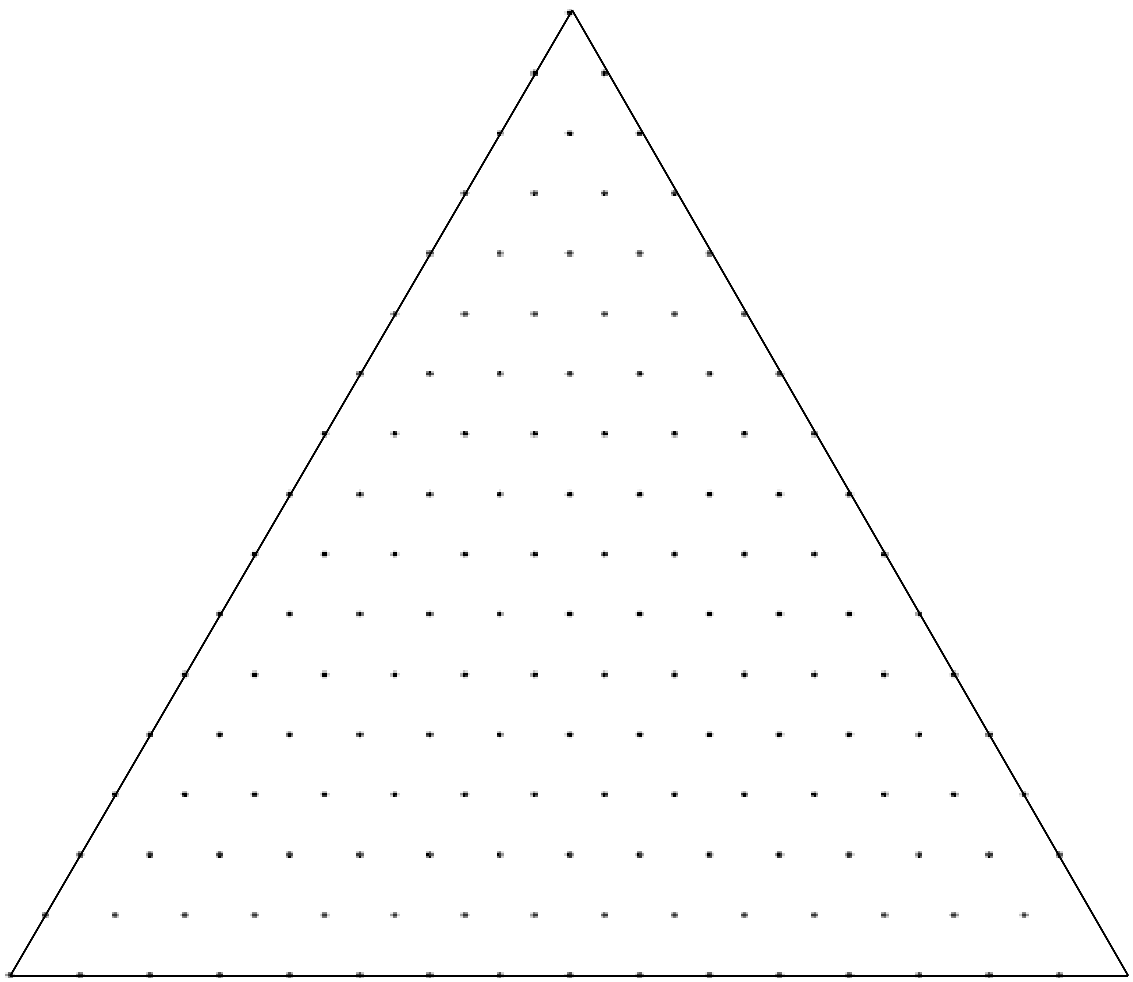
(a) (b) (c)

Stages 3-n: Continue repeating this procedure

EXERCISE 4

Carry out the first four stages in the construction of the Sierpinski Gasket on the triangle shown below. After completing all four stages, shade the remaining gasket.

After each stage of your construction, record the number of triangles, the length of the sides in each gasket, and the area enclosed by the gasket in the chart on the next page.



Perimeter of Gasket

	Number of triangles	Length of each side	Perimeter	Area
Stage 0	1	1	3	$\frac{\sqrt{3}}{4}$
Stage 1	3	$\frac{1}{2}$		
Stage 2				
Stage 3				
Stage 4				
Stage 5				
Stage 6				
Stage 7				
Stage 10				
Stage 100				
Stage n				

- EXERCISE 5: Analyze the patterns which appear in each column in the chart above and then fill in remaining rows of the chart.

- EXERCISE 6: as you continue to iterate this procedure, what appears to be happening…
  (a) To the perimeter?

  ___

  (b) To the area?

  ___

- EXERCISE 7: The actual Sierpinski gasket is the figure obtained when this process is repeated indefinitely
  (a) What is the perimeter of the gasket?

  ___

  (b) What is the area of the gasket?

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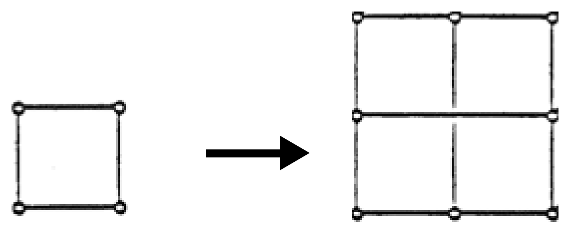
Creating a large Sierpinski Gasket:
You have been constructing a Sierpinski Gasket by “zooming in” and creating smaller and smaller triangles. You can also “zoom out” by pasting your Gasket together with those of several other classmates.

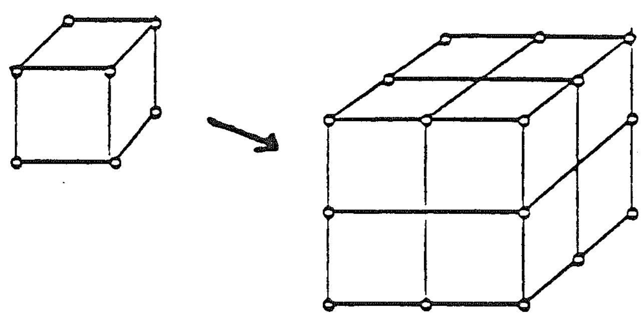
In order to determine the dimension of fractal curves, we need to understand the relation of dimension and scaling factors. We will consider three examples to explore this idea.

(a) Consider a segment whose length is 1 unit. If we double its length, i.e., expand it a scaling factor of 2, we get a new segment which can be divided into two segments, both of which are congruent to the first.

(b) Consider a unit square, each of whose sides is 1 unit. If we expand the square by a scaling factor of 2, i.e., double each of its sides, we get a. new square which can be divided into 4 squares all of which are congruent to the first.



(c) Consider a unit cube. If we expand the cube by a scaling factor of 2, i.e., double each of its sides, we get a new cube which can be divided into 8 cubes all of which are congruent to the first.



- EXERCISE: Assume we can expand a four dimensional unit “box” by a scaling factor of 2. Then, how many boxes are congruent to the first are contained in this new “box”?
  -Give a formula to explain your answer

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These examples are summarized in the chart on the next page. You will need to fill in the information for the four-dimensional unit "box."

Growth Caused by Scaling Factor of 2

Original figure	No. of sides	No. of congruent figures after scaling	Formula
Segment	1	2	$2^1 = 2$
Square	2	4	$2^2 = 4$
Cube	3	8	$2^3 = 8$
4-D Box	4	__	______

- EXERCISE 5: Sketch the following figures and the results of applying a scaling factor of 3 to each

(1) A unit segment

(2) A unit square

(3) A unit cube

- EXERCISE 6: Use your sketches in exercise 5 to fill in the following chart. (Hint: Your formulas should all involve powers of 3

Growth Caused by Scaling Factor of 3

Original figure	Dimension Of Figure	No. of congruent figures after scaling	Formula
Segment			
Square			
Cube			
4-D Box			

- EXERCISE 7: Compare the formulas obtained in exercises 5 and 6. If *N* is the number of figures congruent to the original after scaling a self-similar, *s* is the scaling factor, and *d* is the dimension, then what is the equation relating these three quantities

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SELF-SIMILARITY DIMENSION

The Equation you found in exercise 7 allows us to compute the dimension of a self-similar figure provided we know the number of figures congruent to the original under a given scaling factor. Whenever we compute dimension using this equation, we will refer to the dimension as the *Self-similarity* dimension. In the case of familiar Euclidean figures like segments, squares, and cubes, this equation gives us integers or whole numbers. However, for extremely "wiggly" figures like fractals, we will see that the formula gives us fractions.

WORKSHEET 4: DIMENSION OF FRACTALS

The geometrical figures normally studied in a high school geometry course all have integer (whole number) dimensions. For example, a segment has dimension 1, a square has dimension 2, and a cube has dimension 3. However, fractal curves like the Koch Snow-flake wiggle so much that they partially fill the space between one dimension and the next higher one. Since a fractal is a self-similar figure, we can compute its self-similarity dimension "d" by using the formula developed in Worksheet 3 where N is the number of figures congruent to the original after magnification and s is the scaling factor.

$$(1)\ N = s^d$$

DIMENSION OF THE KOCH SNOWFLAKE

As we saw on Worksheet 2, each stage in the construction of the Koch Snowflake involves replacing a line segment by a broken line consisting of 4 congruent segments, each of which is 1/3 as long as the original.

Hence, if we magnify any side by a scaling factor of $s = 3$ we will have $N = 4$ sides, each of which is congruent to the original side. Using these numbers in the equation $N=s^d$ says that the dimension d of the Koch Snowflake must satisfy the equation $4=3^d$.

Since the dimension d is in the exponent of this equation, we cannot solve for d precisely without using logarithms. However, we can estimate the value of d. It should be obvious that d is between 1 and 2. Exercises 1 and 2 lead to a better estimate for d.

- EXERCISE 1: Use the y^x (*or* x^y) key on your calculator to compute 3^d for each of the values of d in the following chart (round your answers to the nearest hundredth)

Estimating the Dimension of the Koch Snowflake to the Nearest Tenth
By estimating the solution of 3^d

d	3^d	d	3^d	d	3^d
1.0		1.4		1.8	
1.1		1.5		1.9	
1.2		1.6		2.0	
1.3		1.7		XXX	XXX

- EXERCISE 2:

  (A) Use your values in the previous chart to fill in the blank statement below

  $4 = 3^d$ for d between __________ and ___________

  (B) Enter the smaller of these two values as the first value for *d* in the chart below. Then complete the chart to obtain an estimate for the solution of $4 = 3^d$ to the nearest hundredth. Note that the last value of *d* in the chart should be the larger of the two values given in (a).

Estimating the Dimension of the Koch Snowflake to the Nearest Hundredth

By estimating the solution of $4 = 3^d$

d	3^d	*d*	3^d	*d*	3^d
				XXXX	XXXX

(C) Use the chart above to fill in the blank in the statement below

The self-similarity dimension of the Koch Snowflake (to the nearest hundredth) is

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- EXERCISE 3: We have noted that the Koch snowflake is somewhat like a coastline. As we have seen, the self-similarity dimension of the Snowflake is a fraction whose value is between 1 and 2. Mathematicians have used statistical procedures to estimate the self-similarity dimension of coastlines. They have concluded that this dimension is approximately 1.2. Which do you think is more "wiggly", the Koch Snowflake or a coastline? Why?

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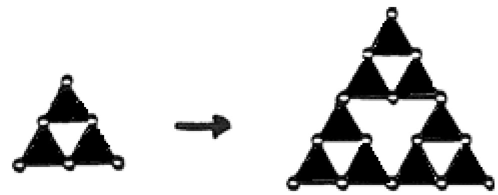
DIMENSION OF THE SIERPINSKI GASKET

In order to compute the self-similarity dimension of the Sierpinski Gasket, we need to determine an appropriate scaling factor. Since the construction of the Gasket involves bisecting the sides of an equilateral triangle, we see that we can use a scaling factor of 2 in order to double the size of the exterior triangle. Once we apply this scaling factor, we obtain 3 congruent copies of the original triangle.



Hence, if we magnify the Sierpinski Gasket by a scaling factor of s = 2 we will have N = 3 copies, each of which is congruent to the original Gasket. Using these numbers in the equation $N = 3^2$ says that the dimension d of the Sierpinski Gasket must satisfy the equation $3 = 2^d$. Again since the dimension d is in the exponent of this equation, we will estimate the value of d. Here too, it should be obvious that d is between 1 and 2. Exercises 4 and 5 lead to a better estimate for d.

• EXERCISE 4: Use the y^x (*or* x^y) key on your calculator to compute 2^d for each of the values of d in the chart below (round your answers to the nearest hundredth).

Estimating the Dimension of the Koch Snowflake to the Nearest Hundredth

By estimating the solution of $4 = 3^d$

d	3^d	d	3^d	d	3^d
1.0		1.4		1.8	
1.1		1.5		1.9	
1.2		1.6		2.0	
1.3		1.7		XXXXXXX	XXXXXXX

- EXERCISE 5:
  (a) Use your values in the previous chart to fill in the blanks in the statement below:

  $3 = 2^d$ for d between __________ and ___________

  (b) Enter the smaller of these two values as the first value for d in the chart on the next page. Then complete the chart to obtain an estimate for the solution of $3 = 2^d$ to the nearest hundredth. Note that the last value of d in the chart should be the larger of the two values even in (a).

Estimating the Dimension of the Koch Snowflake to the Nearest Hundredth
By estimating the solution of 4 = 2^d

d	3^d	*d*	3^d	*d*	3^d
				XXXX	XXXX

(C) Use the chart above to fill in the following blank: The self-similarity dimension of the Sierpinski gasket (to the nearest hundredth) is ______________

• EXERCISE 6: Which of the two fractal curves, the Koch Snowflake and the Sierpinski gasket, is more wiggly? Why?

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-Your answer to Exercise 5c should be somewhat surprising, since in our results on Worksheet 3, we noticed that a scaling factor of 2 applied to common Euclidean figures such as segments, squares and cubes produced 2, 4 or 8 copies, respectively. Since all of these numbers are powers of 2, the self-similarity dimensions of these figures are all integers. Fractals, on the other hand, as their name suggests have fractional self-similarity dimensions.

UNIT SUMMARY:

A fractal is a geometric figure which has three important properties. Fractals are an iterative process, are self similar, and have fractional self-similarity dimension

The new geometry of fractals allows computer generation of pictures which look amazingly like coastlines, mountains, clouds, etc. to create these pictures, computers repeatedly draw a basic shape on smaller and smaller scales. To make the results more realistic, computers are instructed to throw in random variations

However, fractals are not only used to draw realistic pictures. One indication of the number of applications and the recent nature of the study of fractals is their appearance in *Physical Review Letters*, one of the major publications of current physics research. The first paper which used the word fractal appeared in 1980. Now such articles appear in nearly every issue of this journal